Sample: Multivariable Calculus - Basic Definitions and Properties

Here integration and differentiation are performed separately and independently with every coordinate.

1

$$r(t) = \sqrt{t}\vec{i} + 3t\vec{j} - 4t\vec{k}$$

$$||r(t)|| = \sqrt{r(t) \cdot r(t)} = \sqrt{t + 9t^2 + 16t^2} = \sqrt{25t^2 + t}$$

2

The curve is oriented positively. See attachment for the sketch.

3

From the second equation we have

$$y = 4 - x = 2 - \sin t$$

From the third equation we have

$$z = \sqrt{10 - x^2 - y^2} = \sqrt{10 - 4 - 4\sin t - \sin^2 t - 4 + 4\sin t - \sin^2 t} = \sqrt{2 - 2\sin^2 t}$$

Hence,

$$r(t) = (2 + \sin t)\vec{i} + (2 - \sin t)\vec{j} + \sqrt{2 - 2\sin^2 t}\vec{k}$$

4

The fact, that the functions are vector valued, tells us, that we can work with their components independently. That is, for every component we have limit of product of two single-valued function. So, we can use the fact about from limits theory, that the limit of product is the product of limits. Hence

$$\lim_{t\to c}[r(t) \cdot u(t)] = \lim_{t\to c} r(t) \cdot \lim_{t\to c} u(t)$$

5

$$r'(t) = (1/t\vec{i} + 16t\vec{j} + t^2/2\vec{k})' = (-1/t^2\vec{i} + 16\vec{j} + t\vec{k})$$

$$r'(t) = ((\sin t - t\cos t)\vec{i} + (\cos t + t\sin t)\vec{j} + t^2/2\vec{k})' = (t\sin t\vec{i} + t\cos t\vec{j} + 2t\vec{k})$$

6

$$\int (4t^3\vec{i} + 6t\vec{j} - 4\sqrt{t}\vec{k}) = t^4\vec{i} + 3t^2\vec{j} - 8/3t\sqrt{t}\vec{k}$$

7

$$r'(t) = \frac{1}{1+t^2}\vec{i} + 1/t^2\vec{j} + 1/t\vec{k}, \qquad r(1) = 2\vec{i} + 0\vec{j} + 0\vec{k}$$

$$r(t) = (\arctan t + 2 - \frac{\pi}{4})\vec{i} + (1/t - 1)\vec{j} + \ln t$$

8 and 13

$$||r(t)|| = const$$

This means

$$\frac{d||r(t)||}{dt} = 0$$

$$\frac{d\sqrt{x^2+y^2+z^2}}{dt} = \left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)\frac{1}{2\sqrt{x^2+y^2+z^2}} = 0$$

But

$$(dx/dt, dy/dt, dz/dt) = r'(t)$$

So, previous formula can be rewritten as scalar product

$$\frac{d\sqrt{x^2+y^2+z^2}}{dt} = (r(t) \cdot r'(t))/\sqrt{x^2+y^2+z^2} = 0$$

Hence

$$r(t) \cdot r'(t) = 0$$

We can use this fact for problem n.13. Indeed, we know that $a(t) = v'(t)$. So if speed is constant ($||v(t)|| = const$), then

$$v(t) \cdot a(t) = 0$$

which means, that acceleration is perpendicular to velocity.

9

$$r(t) = 4t\vec{i} + 4t\vec{j} + 2t\vec{k}$$

velocity:

$$v(t) = r'(t) = 4\vec{i} + 4\vec{j} + 2\vec{k}$$

speed:

$$||v(t)|| = \sqrt{4^2 + 4^2 + 2^2} = 6$$

acceleration:

$$a(t) = v'(t) = 0$$

10

$$a(t) = -\cos t\vec{i} - \sin t\vec{j} + 0\vec{k}, \qquad v(0) = 0\vec{i} + \vec{j} + \vec{k}, \qquad r(0) = \vec{i} + 0\vec{j} + \vec{k}$$

$$v(t) = \int a(t)dt = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}$$

$$r(t) = \int v(t)dt = \cos t\vec{i} - \sin t\vec{j} + t\vec{k}$$

11

The initial vertical speed of projectile is

$$v_v = v \cdot \sqrt{2}/2 = \sqrt{2} \cdot 4.5m/s$$

Then, its maximum height will be

$$mgh = mv^2/ + 3\,m \cdot mg$$

$$h = 3\,m + \sqrt{2gh}\sqrt{2 \cdot 9.8 \cdot \sqrt{2} \cdot 4.5} \approx 14.17m$$

The time of flight is

$$2v = gt$$

$$t = 2v/g = \sqrt{2} \cdot 9/9.8 \approx 1.3s$$

And range is

$$s = v_h \cdot t = \sqrt{2}/2v \cdot t =$$

12

$$r(t) = b\cos(wt)\vec{i} + b\sin(wt)\vec{j}$$

$$v(t) = -bw\sin(wt)\vec{i} + bw\cos(wt)$$

$$a(t) = -bw^2\cos(wt)\vec{i} - bw^2\sin(wt)$$

$$||a(t)|| = \sqrt{a(t) \cdot a(t)} = \sqrt{b^2w^4\cos^2 t + b^2w^4\sin^2 t} = bw^2$$