Sample: Functional Analysis - Countable and Uncountable Sets

Question 1

Recall that we have defined Lebesgue outer measure as

$$|E|_0 = \inf_{\mathcal{E}} \sum_{B \in \mathcal{E}} |B|$$

Where $\mathcal{E}$ ranges over all countable coverings of E by closed product boxes and |B| is the product of side lengths when B is such a box. Show that the value of the outer measure remains unchanged if the product boxes are taken to be open instead of closed.

Solution.

$$|E|_0 = \inf_{\mathcal{E}} \sum_{B \in \mathcal{E}} |B|$$

Suppose we take open coverings instead of closed.

Let B_0 be particular closed covering of the set E.

$$B_0 = \bigcup_{i=1}^{\infty} C_i$$

where $C_i = [c_1^i, d_1^i] \times [c_2^i, d_2^i] \times \ldots \times [c_n^i, d_n^i]$

$$|B_0| = \sum_{i=1}^{\infty} \prod_{j=1}^{n} (d_j^i - c_j^i)$$

Consider such open covering of E:

$$O_0 = \bigcup_{i=1}^{\infty} O_i$$

$$O_i = (c_1^i - \delta_i, d_1^i + \delta_i) \times (c_2^i - \delta_i, d_2^i + \delta_i) \times \ldots \times (c_n^i - \delta_i, d_n^i + \delta_i)$$

We see that $O_i \supset C_i$, thus $O_0 \supset B_0 \supset E$, so O_0 covers E.

$$|O_i| = \prod_{j=1}^{n} \left(d_j^i - c_j^i + 2\delta_i\right)$$

Such statements hold:

$$|O_i| > |C_i|;\ \lim_{\delta_i \to 0} |O_i| = |C_i|$$

Thus

$$\forall \varepsilon > 0\ \exists \delta_i : |O_i| \le |C_i| + \varepsilon$$

Let's take such δ_i that

$$|O_i| < |C_i| + \frac{\varepsilon}{2^{i+1}}$$

where ε is some fixed positive number.

Then we will get

$$|O_0| = \sum_{i=1}^{\infty} |O_i| \le \sum_{i=1}^{\infty} |C_i| + \frac{\varepsilon}{2^{i+1}} = |B_0| + \varepsilon$$

So for every closed covering of E we built open covering of E such that it's measure differs from closed covering measure by arbitrary $\varepsilon > 0$. So

$$\inf_{\mathcal{E}_1} \sum_{O \in \mathcal{E}_1} |O| \le \inf_{\mathcal{E}} \sum_{B \in \mathcal{E}} |B|$$

where $\mathcal{E}_1$ is the set of all open coverings of E.

On the other hand, for each open covering

$$O = \bigcup_{i=1}^{\infty} O_i$$

$$O_i = \left(c_1^i, d_1^i\right) \times \left(c_2^i, d_2^i\right) \times \dots \times \left(c_n^i, d_n^i\right)$$

We can consider such closed covering:

$$B = \bigcup_{i=1}^{\infty} B_i$$

$$B_i = [c_1^i, d_1^i] \times [c_2^i, d_2^i] \times \dots \times [c_n^i, d_n^i]$$

such that $B \supset O$ and $|B| = |O|$. Thus

$$\inf_{\mathcal{E}_1} \sum_{O \in \mathcal{E}_1} |O| \geq \inf_{\mathcal{E}} \sum_{B \in \mathcal{E}} |B|$$

Finally we get:

$$\inf_{\mathcal{E}_1} \sum_{O \in \mathcal{E}_1} |O| = \inf_{\mathcal{E}} \sum_{B \in \mathcal{E}} |B|$$

So the value of outer measure remains unchanged if the product boxes are taken to be open instead of closed.

Question 2

Consider the following sets in the real line: let $C_0 := [0,1]$, and for $k > 0$, let

$$C_k = \left(\frac{1}{3} C_{k-1}\right) \cup \left(\frac{1}{3} C_{k-1} + \frac{2}{3}\right)$$

where $\alpha S + \beta := \{\alpha x + \beta | x \in S\}$. The set $C = \bigcap_{i=1}^{\infty} C_k$ is called the Cantor middle-thirds set.

Show that C is an uncountable set.

Solution.

Let's consider elements of the set C in terms of their expansion in base 3.

$$C_0 = [0,1]$$

contains the elements $0.*****$ where $*$ is arbitrary number $0,1$ or 2.

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] = [0, 0.1_3] \cup [0.2_3, 1]$$

The set C_1 contains elements that have expansion $0.a***$ where $a = 0\ or\ 2, *= 0,1, or\ 2$.

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$
$$= [0, 0.01_3] \cup [0.02_3, 0.1_3] \cup [0.2_3, 0.21_3] \cup [0.22_3, 1]$$

C_2 contains elements that have $0\ or\ 2$ as their 1-st and 2-nd digits after the dot.

Generally, C_k contains numbers that have first k digits after the dot 0 or 2 (in base 3). This rule holds for all numbers (including border points $0.k_1k_2 \dots k_n 1$ because they can be expressed in such a way:

$$0.0.k_1k_2 \dots k_n 1 = 0.k_1k_2 \dots k_n 0222 \dots$$

$$C = \bigcap_{i=1}^{\infty} C_k$$

So Cantor middle-thirds set consist of numbers from $[0,1]$ that have only 0 and 2 in their expansion in base 3.

Let’s now prove the Cantor set is uncountable.

Let

$$0.k_1k_2, \dots, k_n, \dots \in C$$

Then $k_i \in \{0,2\}$. Now let’s consider a binary number

$$0.a_1a_2 \dots.$$

that was build sing such a rule:

$$a_i = \begin{cases} 0, k_i = 0 \\ 1, k_i = 2 \end{cases}$$

We just built the bijection between the Cantor set and set of all binary numbers from $[0,1]$, which is uncountable. Thus Cantor set is uncountable.

Question 3

a) Show that every open set in R is a countable union of disjoint open intervals.
b) Find an open set in R^2 which may not be written as a countable union of disjoint open rectangles.

Solution.

a)

Let's create such relation on open set O:

$x{\sim}y$ if and only if x and y are covered by the same open interval $I \subset O$ (we consider also semi-infinite and infinite intervals). Clearly, this relation is an equivalence:

- it is reflexive, because every x lies in O with some open neighborhood
- it is symmetric (if x and y are covered by I, then y and x are covered by I)
- it is transitive, (x,y covered by I, y,z covered by J, then I and J intersect and x,z is covered by $I \cup J$)

Then

$$O = \bigcup_{\alpha} I_{\alpha}$$

O is union of disjoint equivalence classes I_{α}. This classes are clearly open intervals. Number of such intervals is no more than countable because every open interval contains unique rational point, but there are countable many rational points.

So the statement is proved.