Sample: Integral Calculus - Delta Integral

Solve the integral

$$I=\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\frac{x}{m^2x-M^2yz}.$$

$$m=0.511,\qquad M=135.$$

Solution.

This integral has delta-function $\delta(x+y+z-1)$. That's why we can eliminate x from integral

$(x+y+z=1\rightarrow x=1-y-z)$. But now we change limits of integration over z ($\int_0^1 dz\rightarrow\int_0^{1-y}dz$):

$$I=\int_0^1 dy\int_0^{1-y}dz\frac{(1-y-z)}{m^2(1-y-z)-M^2yz}.$$

We can use differentiation with respect to the parameter m^2

$$I=\int_0^1 dy\int_0^{1-y}dz\frac{(1-y-z)}{m^2(1-y-z)-M^2yz}=\frac{\partial}{\partial m^2}\int_0^1 dy\int_0^{1-y}dz\ln|m^2(1-y-z)-M^2yz|.$$

Let's make the substitution $z=t(1-y)$:

$$I=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\ln|(1-y)(m^2(1-t)-M^2yt)|.$$

Using that $\ln a\cdot b=\ln a+\ln b$

$$I=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\,(\ln|1-y|+\ln|m^2(1-t)-M^2yt|).$$

Expression $\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\ln|1-y|$ is equal 0, because $\int_0^1 dy\,(1-y)\int_0^1 dt\ln|1-y|$ doesn't depend on m^2.

$$I=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\ln|m^2(1-t)-M^2yt|=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\ln|m^2-(m^2+M^2y)t|.$$

$$I=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\ln|m^2-(m^2+M^2y)t|.$$

We can use that

$$\int dx\ln|A+Bx|=\frac{A}{B}\ln|A+Bx|+x\ln|A+Bx|-x+const$$

$$\int dt\ln|m^2-(m^2+M^2y)t|=$$

$$=\frac{m^2}{-(m^2+M^2y)}\ln|m^2-(m^2+M^2y)t|+t\ln|m^2-(m^2+M^2y)t|-t+const$$

$$\int_0^1 t\ln|m^2-(m^2+M^2y)t|=-\frac{m^2}{(m^2+M^2y)}\ln|m^2-(m^2+M^2y)|+\ln|m^2-(m^2+M^2y)|-1+$$

$$+\frac{m^2}{(m^2+M^2y)}\ln|m^2|$$

$$\int_0^1 t\ln|m^2-(m^2+M^2y)t| = \frac{m^2+M^2y-m^2}{(m^2+M^2y)}\ln|m^2-(m^2+M^2y)|-1+$$

$$+\frac{m^2}{(m^2+M^2y)}\ln|m^2|$$

$$\int_0^1 t\ln|m^2-(m^2+M^2y)t| = \frac{M^2y\ln|M^2y|}{(m^2+M^2y)}-1+$$

$$+\frac{m^2}{(m^2+M^2y)}\ln|m^2|$$

Further

$I=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\int_0^1 dt\,\ln|m^2-(m^2+M^2y)t| = \frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\left(\frac{M^2y\ln|M^2y|}{(m^2+M^2y)}-1+\frac{m^2}{(m^2+M^2y)}\ln|m^2|\right)=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\left(\frac{(M^2y+m^2)\ln|M^2y|}{(m^2+M^2y)}-\frac{m^2\ln|M^2y|}{(m^2+M^2y)}-1+\frac{m^2}{(m^2+M^2y)}\ln|m^2|\right)=$

$$=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\left(\ln|M^2y|-\frac{m^2\ln|M^2y|}{(m^2+M^2y)}-1+\frac{m^2}{(m^2+M^2y)}\ln|m^2|\right)=$$

$$=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\left(\frac{m^2\ln|M^2y|+M^2y\ln|M^2y|-m^2\ln|M^2y|-m^2-M^2y+m^2\ln|m^2|}{(m^2+M^2y)}\right)$$

$$=\frac{\partial}{\partial m^2}\int_0^1 dy\,(1-y)\left(\frac{M^2y\ln|M^2y|-m^2-M^2y+m^2\ln|m^2|}{(m^2+M^2y)}\right)$$

Calculate

$\frac{\partial}{\partial m^2}\frac{M^2y\ln(M^2y)-m^2-M^2y+m^2\ln(m^2)}{(m^2+M^2y)} = -\frac{1}{(m^2+M^2y)^2}\Big(\big((m^2+M^2y)*(-2m+2m\ln(m^2)+2m)\big)-2m(M^2y\ln(M^2y)-m^2-M^2y+m^2\ln(m^2))\Big)=-\frac{1}{(m^2+M^2y)^2}\big((2mM^2y\ln(m^2)-2mM^2y\ln(M^2y)+2m^3+2mM^2y)\big)=$

Finally

$-\int_0^1(1-y)*\frac{1}{(m^2+M^2y)^2}\big((2mM^2y\ln(m^2)-2mM^2y\ln(M^2y)+2m^3+2mM^2y)\big)dy=$

We know that $m=0.511$ and $M=135$.

$-\int_0^1(1-y)*\frac{1}{(0.261121+18225y)^2}\big((2*0.511*18225*y\ln(0.261121)-18225y\ln(18225y)+2*0.133432831+2*0,511*18225*y)\big)dy=$

Mathematica gives the result ranged from -0.00272 432 to 0.256654.