Sample: Differential Calculus Equations - Differential Calculus Assignment

1. A thermometer reading 70°F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110°F after $\frac{1}{2}$ minute and 145°F after 1 minute. How hot is oven? Please show all your work including setting up your differential equations.

Solution:

If T is temperature, then by Newton's law

$$\frac{dT}{dt} = -k(T - T_0)$$

Where

T_o – is temperature of oven

Integrate left side of equation from 70°F to 110°F and right side from 0 to $\frac{1}{2}$

$$\int_{70}^{110} \frac{dT}{T - T_o} = \int_{0}^{1/2} -kdt$$

$$\ln(T - T_o)|_{70}^{110} = -kt|_0^{1/2}$$

$$\ln(110 - T_o) - \ln(70 - T_o) = -k\left(\frac{1}{2} - 0\right)$$

$$2\ln\frac{110 - T_o}{70 - T_o} = -k$$

$$k = -2\ln\frac{T_o - 110}{T_o - 70} \ (1)$$

Integrate left side of equation from 110°F to 145°F and right side from $\frac{1}{2}$ to 1

$$\int_{110}^{145} \frac{dT}{T - T_o} = \int_{1/2}^{1} -kdt$$

$$\ln(T - T_o)|_{110}^{145} = -kt|_{\frac{1}{2}}^{1}$$

$$\ln(145 - T_o) - \ln(110 - T_o) = -k\left(1 - \frac{1}{2}\right)$$

$$2\ln\frac{145-T_o}{110-T_o}=-k$$

$$k=-2\ln\frac{T_o-145}{T_o-110}\ (2)$$

Equate (1) to (2)

$$-2\ln\frac{T_o-110}{T_o-70}=-2\ln\frac{T_o-145}{T_o-110}$$

$$\frac{T_o-110}{T_o-70}=\frac{T_o-145}{T_o-110}$$

$$(T_o-110)^2=(T_o-70)(T_o-145)$$

$$T_o^2-220T_o+12100=T_o^2-215T_o+10150$$

$$5T_o=1950$$

$$T_o=390°\mathrm{F}$$

Answer: the temperature of oven is $T_o=390°\mathrm{F}$

2. Solve the given initial-value problem.

$$y'''+2y''-5y'-6y=0$$

$$y(0)=y'(0)=0, y''(0)=1$$

Solution:

Write down the characteristic equation for differential equation and solve it:

$$r^3+2r^2-5r-6=0$$

$$r_1=-3, r_2=-1, r_3=2$$

So, the general solution will be

$$y=C_1e^{-3x}+C_2e^{-x}+C_3e^{2x}$$

To find initial-value problem need to find coefficients C_1, C_2, C_3

Find first and second derivatives of general solution

$$y'=-3C_1e^{-3x}-C_2e^{-x}+2C_3e^{2x}$$

$$y'' = 9C_1e^{-3x} + C_2e^{-x} + 4C_3e^{2x}$$

Plug in the initial conditions

$$y(0) = 0 = C_1 + C_2 + C_3$$

$$y'(0) = 0 = -3C_1 - C_2 + 2C_3$$

$$y''(0) = 1 = 9C_1 + C_2 + 4C_3$$

Solve a system of three equations and three unknowns:

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -3C_1 - C_2 + 2C_3 = 0 \\ 9C_1 + C_2 + 4C_3 = 1 \end{cases}$$

Add first and second equations and second and third equations:

$$\begin{cases} -2C_1 + 3C_3 = 0 \\ -3C_1 - C_2 + 2C_3 = 0 \\ 6C_1 + 6C_3 = 1 \end{cases}$$

Multiply first equation by 3 and add it to third:

$$\begin{cases} 15C_3 = 1 \\ -3C_1 - C_2 + 2C_3 = 0 \\ 6C_1 + 6C_3 = 1 \end{cases}$$

So,

$$C_3 = \frac{1}{15}$$

$$C_1 = \frac{1}{6} - C_3 = \frac{1}{6} - \frac{1}{15} = \frac{1}{10}$$

$$C_2 = -C_1 - C_3 = -\frac{1}{15} - \frac{1}{10} = -\frac{1}{6}$$

The solution to the initial-value problem is then,

$$y = \frac{1}{10}e^{-3x} - \frac{1}{6}e^{-x} + \frac{1}{15}e^{2x}$$

Answer: $y = \frac{1}{10}e^{-3x} - \frac{1}{6}e^{-x} + \frac{1}{15}e^{2x}$

3. Solve the given differential equation by undetermined coefficients.

$$y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$$

Solution:

The general solution will be of the form,

$$y(x) = y_c(x) + Y_p(x)$$

Where

$y_c(x) -$ is the complementary solution

$Y_p(x) -$ is the particular solution.

Complementary solution comes from solving,

$$y''' - y'' - 4y' + 4y = 0$$

Write down the characteristic equation for differential equation and solve it,

$$r^3 - r^2 - 4r + 4 = 0$$

$$r_1 = -2, r_2 = 1, r_3 = 2$$

So, the general solution will be

$$y = C_1 e^{-2x} + C_2 e^x + C_3 e^{2x}$$

The particular solution will be

$$Y_p(x) = A + Bxe^x + Cxe^{2x}$$

To find coefficients A, B, C need to differentiate particular solution and plug into the differential equation,

$$Y_p' = Be^x + Bxe^x + Ce^{2x} + 2Cxe^{2x} = Be^x(1 + x) + Ce^{2x}(1 + 2x)$$

$$Y_p'' = Be^x(1 + x) + Be^x + 2Ce^{2x}(1 + 2x) + 2Ce^{2x} = Be^x(2 + x) + 4Ce^{2x}(1 + x)$$

$$Y_p''' = Be^x(3 + x) + 4Ce^{2x}(3 + 2x)$$

Plug derivatives into the differential equation

$$Be^x(3 + x) + 4Ce^{2x}(3 + 2x) - \left(Be^x(2 + x) + 4Ce^{2x}(1 + x)\right) - 4\left(Be^x(1 + x) + Ce^{2x}(1 + 2x)\right) + 4(A + Bxe^x + Cxe^{2x}) = 5 - e^x + e^{2x}$$

$$4A + e^x(3B + Bx - 2B - Bx - 4B - 4Bx + 4Bx) + e^{2x}(12C + 8Cx - 4C - 4Cx - 4C - 8Cx + 4Cx) = 5 - e^x + e^{2x}$$

$$4A - 3Be^x + 4Ce^{2x} = 5 - e^x + e^{2x}$$

So,

$$A=\frac{5}{4}, B=\frac{1}{3}, C=\frac{1}{4}$$

$$Y_p(x)=\frac{5}{4}+\frac{x}{3}e^x+\frac{x}{4}e^{2x}$$

Therefore solution will be

$$y(x)=C_1e^{-2x}+C_2e^x+C_3e^{2x}+\frac{5}{4}+\frac{x}{3}e^x+\frac{x}{4}e^{2x}$$

Answer: $y(x)=C_1e^{-2x}+C_2e^x+C_3e^{2x}+\frac{5}{4}+\frac{x}{3}e^x+\frac{x}{4}e^{2x}$