Sample: Abstract Algebra - Groups

1)

Suppose f is bijection. Then for each $x \in X$ exists unique $y \in X$ such that $f(y) = x$. This y depends on x, so let $y = g(x)$. So we have $f\big(g(x)\big) = x$ for all x. Let's prove $g\big(f(x)\big) = x$. Substituting $x \to f(x)$ we get:

$$f\Big(g\big(f(x)\big)\Big) = f(x)$$

Since f is bijection this implies $g\big(f(x)\big) = x$ for all x.

Now suppose there exists g such that $f\big(g(x)\big) = g\big(f(x)\big) = x$. We need to prove f is bijection.

Since $f\big(g(x)\big) = x$ for all x, then range of function f equals to X (because for every $x \in X$ exists $y = g(x)$ such that $f(y) = x$). So f is surjective.

Let's prove f is injective. Suppose $\exists x_1 \neq x_2 : f(x_1) = f(x_2)$. Then

$$g\big(f(x_1)\big) = x_1; g\big(f(x_2)\big) = x_2$$

This contradicts to the fact $g\big(f(x_1)\big) = g\big(f(x_2)\big)$. So f is injective.

SO f is surjective and injective, so f is bijection.

2)

Suppose a has 2 inverses b_1 and b_2, e is unit element. Then

$$b_1 a = a b_1 = e$$

$$b_2 a = a b_2 = e$$

Let's take the equality $a b_1 = e$ and multiply it by b_2 from the left. We get:

$$b_2 a b_1 = b_2$$

Since $b_2 a = e$ we get:

$$b_2ab_1 = eb_1 = b_1 = b_2$$

We get contradiction with the fact $b_1 \neq b_2$.

3)

Suppose there exists 2 different identity elements e_1 and e_2. Then

$$e_1e_2 = (since\ e_1\ is\ identity\ element) = e_2$$

$$e_1e_2 = (since\ e_2 is\ identity\ element) = e_1$$

So $e_1 = e_2$. We got contradiction. So there is only one identity element.

4)

We need to prove S^* that consist of all invertible elements is a group.

Firstly note, that identity element e belongs to S^* because $e^{-1} = e$. Next, if $x \in S^*$ then $x^{-1} \in S^*$, because $(x^{-1})^{-1} = x$ (the last inequality follows from $xx^{-1} = x^{-1}x = e$). So every element in S^* has inverse element.S^* is closed under law of composition from S, because if $a, b \in S^*$, element ab has inverse $b^{-1}a^{-1}$, because

$$abb^{-1}a^{-1} = (associativity) = a(bb^{-1})a^{-1} = aea^{-1} = aea^{-1} = e$$

Associativity of composition holds in S^* because it holds in S.

So S^* is a group.