Sample: Differential Calculus Equations - Mathematical Modelling

1. Solve the following initial-value problems. For (a) and (b) show by substitution that your solutions are correct (both the equation and the initial condition - to get full marks you MUST do this). Of course you should ALWAYS check your solutions.

(a) $y' = \frac{1}{2}y^2 e^{x/3}, y(0) = 1/2.$

(b) $y' - \frac{2}{x}y = \frac{1}{2}x + 2, y(1) = -2.$

(c) $\frac{dw}{dt} = \pi w^{-2} \sin 3t\,, w(0) = 1.$

Solution

(a) $y' = \frac{1}{2}y^2 e^{x/3}, y(0) = 1/2$

This is differential equation of the first order, with separable variables. Separate variables:

$$\frac{dy}{y^2} = \frac{1}{2}e^{x/3}dx$$

Integrate both sides of equation:

$$\int \frac{dy}{y^2} = \frac{1}{2}\int e^{x/3}dx$$

Then we obtain the general integral of equation:

$-\frac{1}{y} = \frac{3}{2} * e^{\frac{x}{3}} + C$, where C is an arbitrary real constant.

Solve the initial-value problem.

Solution satisfies $y(0) = 1/2$, that is why $-2 = \frac{3}{2} + C$, and $C = -\frac{7}{2}$,

Finally

$y(x) = -\frac{2}{3e^{x/3}-7}$ is the solution to initial-value problem.

And check up the given function: $y(0) = -\frac{2}{3-7} = -\frac{2}{-4} = \frac{1}{2}$

Calculate $y' = -2\left(-\frac{1}{\left(3e^{\frac{x}{2}}-7\right)^2}\right) * 3e^{\frac{x}{2}} * \frac{1}{3} = \frac{1}{2}y^2 e^{x/3}$. We obtained the correct solution.

(b) $y' - \frac{2}{x}y = \frac{x}{2} + 2,$

$y(1) = -2.$

This is differential equation of the first order, linear differential equation. Substitute

$y = uv$ (1)

Rewrite equation in terms of u and : $u'v + uv' - \frac{2}{x}uv = \frac{x}{2} + 2,$

$u'v + u\left(v' - \frac{2}{x}v\right) = \frac{x}{2} + 2.$ (2)

Suppose

$v' - \frac{2}{x}v = 0,$ (3)

$v' = \frac{2}{x}v, \frac{dv}{v} = \frac{2dx}{x},$ by integrating this equation we obtain

$$ln|v| = 2ln|x| + lnC$$

According to properties of logarithm $|v| = Cx^2$. Chosing the constant C we can rewrite $v = Cx^2$. We select a single solution

$v = x^2$ (4)

From (3), equation (2) becomes $u'v = \frac{x}{2} + 2$, where we seek v in the form (4).

So we arrive at the following equation

$u'x^2 = \frac{x}{2} + 2$, or $u' = \frac{1}{2x} + \frac{2}{x^2}$, integrating the last equation we obtain

$u = \frac{1}{2} ln|x| - \frac{2}{x} + C_1$. (5)

We recall (1), (4), (5) and get

$y(x) = \left(\frac{1}{2} ln|x| - \frac{2}{x} + C_1\right) x^2$ (6)

Solve the initial-value problem. Solution satisfies condition

$y(1) = -2$, that is why from (6) we have

$y(1) = \left(\frac{1}{2} ln|1| - \frac{2}{1} + C_1\right) 1 = C_1 - 2 = -2$.

And $C_1 = 0$.

Finally $y(x) = \left(\frac{x^2}{2} ln|x| - 2x\right)$ is the solution to initial-value problem.

Check up the given function: $y(1) = \frac{1}{2} ln1 - 2 = -2$,

For $x > 0$

$$y'(x) = \left(\frac{x^2}{2} ln|x| - 2x\right)' = \left(\frac{x^2}{2} lnx - 2x\right)' = xlnx + \frac{x}{2} - 2$$

For $x < 0$

$$y'(x) = \left(\frac{x^2}{2} ln|x| - 2x\right)' = \left(\frac{x^2}{2} \ln(-x) - 2x\right)' = xln(-x) + \frac{x}{2} - 2$$

In fact $y'(x) = xln|x| + \frac{x}{2} - 2$

So $y'(x) - \frac{2}{x} y(x) = xln|x| + \frac{x}{2} - 2 - \frac{2}{x}\left(\frac{x^2}{2} ln|x| - 2x\right) = \frac{x}{2} + 2$. We obtained the correct solution.

(c) $\frac{d\omega}{dt} = \pi\omega^{-2} sin3t, \omega(0) = 1$.

This is differential equation of the first order, with separable variables. Separate variables: $\omega^2 d\omega = \pi sin3tdt$.

Integrate both sides of equation:

$\int \omega^2 d\omega = \int \pi sin3tdt$.

Then we obtain the general integral of equation:

$\frac{\omega^3}{3} = \pi * \left(-\frac{1}{3}\right) cos3t + C$, where C is an arbitrary real constant. Solve the initial-value problem.

Solution satisfies $\omega(0) = 1$, that is why $\frac{1}{3} = C - \frac{\pi}{3}$, and $C = \frac{\pi+1}{3}$

Finally we obtain:

$\omega = \sqrt[3]{\pi + 1 - \pi cos3t}$ is the solution to initial-value problem.

2. A simple model for the absorption of a herbicide placed on the surface of a leaf is that it will be absorbed at a rate proportional to the difference in concentration between that on the surface and that on the interior. Assume the rate constant is α.
(a) If a layer with constant concentration C_a is placed on the leaf, write an equation for the concentration of herbicide in the leaf, C(t) as time progresses.
(b) Solve the equation for concentration C(t) if there is no herbicide in the leaf to begin.

Solution.

(a) $C'(t) = \alpha(C_0 - C(t))$

This is differential equation of the first order, linear differential equation. We neglect the influence of the reverse chemical reaction.

(b) We know that general solution to linear inhomogeneous equation is the sum of general solution to homogeneous equation and particular solution to linear inhomogeneous equation.

If we substitute expression into the equation it is obvious that $C(t) = C_0$ is a particular solution of linear inhomogeneous equation.

Homogeneous linear equation is the following:

$$C'(t) + \alpha C(t) = 0$$

which has the solution $C(t) = C_1 e^{-\alpha t}$.

So $C(t) = C_1 e^{-\alpha t} + C_0$ is the general solution to linear inhomogeneous equation.

3. Suppose the equation for the rate of change of substrate in an enzyme kinetics reaction is approximately given by

$$\frac{ds}{dt} = 0.2s - \frac{1.8s^2}{s+1},$$
$$s(0) = 1,$$

where s(t) is the concentration of substrate.
Compute any equilibrium points and test their stability. Suggest the final value of s(t) as t → ∞.

Solution.

Function $f(s) = 0.2s - \frac{1.8s^2}{s+1}$ equals to zero 0 at points $s_1 = 0$ and $s_2 = \frac{1}{8}$. That is why $s = 0$ and $s = \frac{1}{8}$ are equilibrium points.

Besides $f(s) > 0$ under $0 < s < \frac{1}{8}$; $f(s) < 0$ under $s < 0$ or $s > \frac{1}{8}$. Solutions that start "near" $s = 0$ all move away from it as t increases and so $s = 0$ is an unstable equilibrium solution. Solutions that start "near" $s = \frac{1}{8}$ all move towards it as t increases and so $s = \frac{1}{8}$ is an asymptotically stable equilibrium solution ($s(t) \to \frac{1}{8}$ as $t \to +\infty$).