# **Sample:** Matrix Tensor Analysis - Matrix Problems

**Q1.** (35 marks) Let

$$\mathbf{a} = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix}$$

a. Find a basis for the column space Col($\mathbf{a}$).

b. Find a basis for the row space Row($\mathbf{a}$).

c. Find a basis for its null space Null($\mathbf{a}$).

d. Which of the following vectors are in Row($\mathbf{a}$): $[2,-3,1,-1]$, $[2,2,-1,4]$, $[1,4,6,1]$, $[-3,1,-5,-2]$.

e. Find all vectors in Row($\mathbf{a}$) with the first two components equal (ie, all vectors of the form $[a,a,b,c]$ for some real numbers $a,b,c$).

## SOLUTION Q1 (a)

To determine which columns of the matrix form the basis of the column space we perform elementary row operations to receive reduced row echelon form (elementary row operations do not affect the dependence relations between columns).

$$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 4 & 4 & 4 \\ 0 & -3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & -3 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we can see that the first and the second columns form a basis of the column space (we say this way since these columns have leading ones in reduced row echelon form).

Thus,

$$Col(a) = span < \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} >$$

## SOLUTION Q1 (b)

To determine which rows of the matrix form the basis of the row space we perform elementary row operations to receive row echelon form (elementary row operations do not affect the row space). This was done in part (a):

$$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-zero rows in row echelon form are in basis of row space:

$$Row(a) =< \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} >$$

## SOLUTION Q1 (c)

The null space is orthogonal to the row space. To find it we should solve the system $ax = 0$:

$$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, in fact, we can solve the same system for $a$ in reduced row echelon form, which is easier:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x_1 + 2x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -2x_3 - x_4 \\ x_2 = -x_3 - x_4 \\ x_3, x_4 - free \end{cases}$$

Thus, vectors in the null space have the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, where $x_3$ and $x_3$ are arbitrary values.

Therefore $Ker(a) =< \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} >$

## SOLUTION Q1 (d)

From the part (b) we can see that all vectors in the row space have the form $[a, b, (2a+b), (a+b)]$, where $a, b$ are arbitrary constants. Really,

$$ar_1 + br_2 = a[1,0,2,1] + b[0,1,1,1] = [a, b, (2a+b), (a+b)]$$

Therefore to check if the vector $[x_1, x_2, x_3, x_4]$ is in the row space we should simply check if $2x_1 + x_2 = x_3$ and $x_1 + x_2 = x_4$.

Let's do that:

1) $[x_1, x_2, x_3, x_4] = [2,-3,1,-1]$,
   $2x_1 + x_2 = 2 \cdot 2 - 3 = 1 = x_3$, $x_1 + x_2 = 2 - 3 = -1 = x_4$. This vector is in the row space.
2) $[x_1, x_2, x_3, x_4] = [2,2,-1,4]$,
   $2x_1 + x_2 = 2 \cdot 2 + 2 = 6 \neq x_3$. This vector is not in the row space.
3) $[x_1, x_2, x_3, x_4] = [1,4,6,1]$,
   $x_1 + x_2 = 1 + 4 = 5 \neq x_4$. This vector is not in the row space.
4) $[x_1, x_2, x_3, x_4] = [-3,1,-5,-2]$,
   $2x_1 + x_2 = 2 \cdot (-3) + 1 = -5 = x_3$, $x_1 + x_2 = -3 + 1 = -2 = x_4$. This vector is in the row space.

**SOLUTION Q1 (e)**

From the part (c) we can see that all vectors in the row space have the form $[a, b, (2a+b), (a+b)]$, where $a, b$ are arbitrary constants. Now let two first components be same, then

$$[a,a,b,c] = [x_1, x_2, (2x_1 + x_2), (x_1 + x_2)] \to \begin{cases} a = x_1 \\ a = x_2 \\ b = x_3 = 2x_1 + x_2 \\ c = x_4 = x_1 + x_2 \end{cases} \to \begin{cases} x_1 = a \\ x_2 = a \\ x_3 = 2a + a = 3a \\ x_4 = a + a = 2a \end{cases}$$

Thus, all such vectors have the form $[x_1, x_2, x_3, x_4] = [a, a, 3a, 2a]$, where $a$ is arbitrary value. This set can be written as the following span:

$$\{[a, a, 3a, 2a] \ | a \in R\} = \{a[1,1,3,2] \ | a \in R\} = span < [1,1,3,2] >$$

**Q1.** (35 marks) Let

$$\mathbf{a} = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix}$$

a. Find a basis for the column space $\mathrm{Col}(\mathbf{a})$.

b. Find a basis for the row space $\mathrm{Row}(\mathbf{a})$.

c. Find a basis for its null space $\mathrm{Null}(\mathbf{a})$.

d. Which of the following vectors are in $\mathrm{Row}(\mathbf{a})$: $[2,-3,1,-1]$, $[2,2,-1,4]$, $[1,4,6,1]$, $[-3,1,-5,-2]$.

e. Find all vectors in $\mathrm{Row}(\mathbf{a})$ with the first two components equal (ie, all vectors of the form $[a,a,b,c]$ for some real numbers $a,b,c$).

**SOLUTION Q1 (a)**

To determine which columns of the matrix form the basis of the column space we perform elementary row operations to receive reduced row echelon form (elementary row operations do not affect the dependence relations between columns).

$$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 4 & 4 & 4 \\ 0 & -3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & -3 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we can see that the first and the second columns form a basis of the column space (we say this way since these columns have leading ones in reduced row echelon form).

Thus,

$$Col(a) = span < \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} >$$

**SOLUTION Q1 (b)**

To determine which rows of the matrix form the basis of the row space we perform elementary row operations to receive row echelon form (elementary row operations do not affect the row space). This was done in part (a):

$$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ -2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q=\begin{bmatrix}0&2&-1\\1&4&-3\\1&-3&2\\2&-1&1\end{bmatrix}\sim\begin{bmatrix}-1&2&0\\-3&4&1\\2&-3&1\\1&-1&2\end{bmatrix}\sim\begin{bmatrix}1&2&0\\3&4&1\\-2&-3&1\\-1&-1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0\\3&-2&1\\-2&1&1\\-1&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0\\3&1&-2\\-2&1&1\\-1&2&1\end{bmatrix}\sim$$

$$\sim\begin{bmatrix}1&0&0\\0&1&0\\-5&1&3\\-7&2&5\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\-5&1&1\\-7&2&5/3\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\4/3&1/3&5/3\end{bmatrix}$$

Then

$$Col(Q)=Span\left(\begin{bmatrix}1\\0\\0\\4/3\end{bmatrix},\begin{bmatrix}0\\1\\0\\1/3\end{bmatrix},\begin{bmatrix}0\\0\\1\\5/3\end{bmatrix}\right)=Col(P)$$

As we see column spaces of both transforms coincide, therefore the previous spans are equal too:

$$Span(p_1,p_2,p_3)=Span(q_1,q_2,q_3)$$