Sample: Discrete Mathematics - Properties of Relations

Task 1. *Suppose we are given set* $A = \{1,2,3,4,6,12\}$ *and a relation,* R*, from* $A \times A$*. The relation is defined as follows:*

$$R = \{(a,b) | \text{ a divides b, } where \text{(a, b) belongs to } A \times A\}.$$

a) List all the ordered pairs (a,b) *that are elements of the relation.*
b) Use the results from part a) to construct the corresponding zero-one matrix.

Solution. a) Relation R consists of the following pairs:

$$(1,1), (1,2), (1,3), (1,4), (1,6), (1,12),$$
$$(2,2), (2,4), (2,6), (2,12), \quad (3,3), (3,6), (3,12)$$
$$(4,4), (4,12), \quad (6,6), (6,12), \quad (12,12)$$

b) The matrix representing this relation has the following form:

	1	2	3	4	6	12
1	1	1	1	1	1	1
2	0	1	0	1	1	1
3	0	0	1	0	1	1
4	0	0	0	1	0	1
6	0	0	0	0	1	1
12	0	0	0	0	0	1

Task 2. *Let* R *be the relation on the set of all people who have visited a particular Web page such that* xRy *if and only if person* x *and person* y *have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that* R *is an equivalence relation (i.e. it is reflexive, symmetric, and transitive).*

Solution.

1) R *is reflexive*, that is xRx for all persons x.
Indeed, x and x have followed the same set of links starting at this Web page.

2) R *is symmetric*, that is xRy then yRx.
Indeed if x and y have followed the same set of links starting at this Web page, then y and z have followed the same set of links starting at this Web page.

3) R *is transitive*, that is if xRy and yRz then xRz.
Indeed if x and y have followed the same set of links starting at this Web page, and y and z have followed the same set of links starting at this Web page, then x and z also have followed the same set of links starting at this Web page.

Task 3. *For the relation* R *on the set* $A = \{1,2,3,4\}$ *given below, deteremine whether it is reflexive, symmetric, anti-symmetric and transitive.*

$$R = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$$

Solution.

1) R is **not reflexive**, since $(1,1) \not\subset R$.
2) R is **not symmetric**, since $(2,3) \in R$ but $(3,2) \not\subset R$.

3) R is **not anti-symmetric**, since both $(1,3)$ and $(3,1)$ belong to R.
4) R is **not transitive**, since $(1,3),(3,1) \in R$ but $(1,1) \notin R$.

Task 4 . *List the ordered pairs in the relation* R *from* $A = \{0,1,2,3,4\}$ *to* $B = \{0,1,2,3\}$, *where* $(a,b) \in R$ *if and only if* $a + b = 4$.
Solution. The relation R consists of the following pairs:

$$(1,3),\ (2,2),\ (3,1),\ (4,0).$$

Task 5. *A department manager has* 4 *employees involved with* 6 *projects throughout the fiscal year. In how many ways can the manager assign these projects so that each employee is working on at least one project?*
Solution. Let $P = \{1,2,3,4,5,6\}$ be the set of all projects. Each assignment is a partition of P into 4 non-empty subsets.
Therefore we should compute the number of partitions of P into ordered family of 4 non-empty subsets A_1, A_2, A_3, A_4.
Since every A_i is non-empty then there possible two distinct cases:
Case 1). One of these sets consists of 3 elements, and each of other three sets consists of a unique element, e.g.

$$A_1 = \{1,2,3\},\quad A_2 = \{4\},\quad A_3 = \{5\},\quad A_4 = \{6\}.$$

Each such partition is determined by three one-element sets, i.e. by choice of ordered 3-tuple from P, and then by 4 positions of the set of remained three elements of P.
Then the number of partitions in this case is equal to

$$6 * 5 * 4 * 4 = 480.$$

Case 2). Two sets are two-elements and other two sets are one-elements, e.g.

$$A_1 = \{1,2\},\quad A_2 = \{3,4\},\quad A_3 = \{5\},\quad A_4 = \{6\}.$$

First let us compute the number of partitions of P into sets of 1, 1, 2 and 2 elements. Indeed, the first 1-elements set can be chosen from 6 elements. To each choice of that elemetns set correspond 5 choices of the second 1-elements set. Then third 2-elements set is chosen from the remained 4 elements, so we can choose that set into $C_4^2 = \frac{4!}{2!*(4-2)!} = 6$ ways. The fourth 2-elements set is then also determined. Hence the number partitions of P into sets of 1, 1, 2 and 2 elements is equal to

$$6 * 5 * C_4^2 = 6 * 5 * 6 = 180$$

Now we should take compute the number of distinct ``words'' obtained by permutations of 1122. This number is equal to the number of choices of two elements (say 1's) from 4-elements set, and so it is 6.
Hence the total number of functions in the case 2 is

$$180 * 6 = 1080.$$

Therefore the total number of all ways that the manager can assign these projects so that each employee is working on at least one project is equal to

$$480 + 1080 = 1560.$$

Task 6. *A survey of households in the U.S. reveals that 96% have at least one television set, 98% have cell phone service and 95% have a cell phone and at least one television set. What percentage of households in the U.S. has neither cell phone nor a television set?*

Solution. Let H be the set of all households, T be the set of all households having at least one television set, and C is the set of all households having cell phone service. For a subset $A \subset X$ denote by $|A|$ the number of elements in A. Then by assumption

$$|T| = 0.96|X|, \qquad |C| = 0.98|X|, \qquad |T \cap C| = 0.95|X|.$$

We should find the percentage of the set

$$X \backslash (T \cup C)$$

in X.

Notice that

$$C \backslash T = C \backslash (T \cap C) = |C| - |T \cap C| = (0.98 - 0.95)|X| = 0.03|X|.$$

Thus the number of households having cell phone service but not television set constitutes 3% over all households.

Hence

$$T \cup C = T \cup (C \backslash T) = (0.96 + 0.03)|X| = 0.99|X|,$$

so the number of households having at least one television set or cell phone service constitute 99% over all households.

Therefore percentage of households in the U.S. has neither cell phone nor a television set is

$$100 - 99 = 1\%.$$

Task 7. *Let* $A = \{a, b, c, d\}$ *and* $B = \{1,2,3,4\}$.

a) How many functions $f: A \to B$ *are there?*

b) How many functions $f: A \to B$ satisfy $f(a) = 2$?

Solution. a) Notice that a function $f: A \to B$ can associate to each of 4 elements A one of the 4 elements of B. To each of 4 choices of $a \mapsto f(a)$, correspond 4 choices of $b \mapsto f(b)$, so we get $4 * 4 = 4^2$ choices of values for a and b.

To each choice of $(f(a), f(b))$ correspond 4 choices of $f(c)$, so we get $4 * 4 * 4 = 4^3$ choices of values for a, b, and c.

Similarly, there are 4^4 choices of values $f(a), f(b), f(c), f(d)$. Thus the number of functions $f: A \to B$ is equal to $4^4 = 256$.

b) A function $f: A \to B$ satisfying $f(a) = 2$, is uniquely determined by its values at points b, c, d. In other words, the number of functions $f: A \to B$ with $f(a) = 2$ is equal to the number of functions $g: \{b, c, d\} \to B$. Similarly to a) the number of such functions $g: \{b, c, d\} \to B$ is equal to $4^3 = 64$.

Task 8. *Use mathematical induction to prove the following proposition:*

$$P(n): 3+5+7+\cdots+2n+1 = n(2+n)$$

where $n = 1,2,3,\ldots$

Proof. Let $n = 1$. Then

$$P(1) = 3,$$

and

$$n(2+n) = 1*(2+1) = 3 = P(1).$$

Suppose that we proved that

$$3+5+7+\cdots+2k+1 = k(2+k)$$

for all $k \le n$. Let us prove this for $k = n+1$, that is

$$P(n): 3+5+7+\cdots+2(n+1)+1 = (n+1)(2+n+1) = (n+1)(n+3) = n^2+4n+3.$$

We have that

$$\begin{aligned} P(n+1) &= 3+5+7+\cdots+2n+1+2(n+1)+1 \\ &= (3+5+7+\cdots+2n+1)+2(n+1)+1 \\ &= n(2+n)+2(n+1)+1 \\ &= 2n+n^2+2n+2+1 \\ &= n^2+4n+3. \end{aligned}$$

Now by induction relation

$$P(n): 3+5+7+\cdots+2n+1 = n(2+n)$$

holds for all $n \ge 1$.

Task 9. *Express the greatest common divisor of the following pair of integers as a linear combination of the integers:*

$$117, \quad 213$$

Solution. First we find prime decompositions of 117 and 213:

$$117 = 3*39 = 3*3*13 = 3^2*13,$$
$$213 = 3*71.$$

Thus

$$GCD(117,213) = 3.$$

We should find numbers p and q such that

$$117p + 213q = 3.$$

Contracting by 3 we obtain

$$39p + 71q = 1.$$

Notice that this identity modulo 39 and 71 means that

$$71q \equiv 1\text{mod}39, \qquad 39p \equiv 1\text{mod}71.$$

Let $\phi(m)$ be the Euler function which is equal to the number of numbers a such that $1 \le a < m$ and $GCD(a,m) = 1$. Then by Euler theorem if $GCD(b,m) = 1$, then

$$b^{\phi(m)} \equiv 1\text{mod}m.$$

It is known that

$$\phi(p) = p - 1$$

for any prime p, and if $GCD(a,b) = 1$, then

$$\phi(a*b) = \phi(a)*\phi(b).$$

Hence

$$\phi(71) = 71 - 1 = 70,$$

and

$$\phi(39) = \phi(3*13) = \phi(3)*\phi(13) = (3-1)*(13-1) = 2*12 = 24.$$

Since $GCD(71,39) = 1$, it follows from Euler theorem that

$$71^{\phi(39)} = 71^{24} = 1\text{mod}39$$

$$39^{\phi(71)} = 39^{70} = 1\text{mod}71.$$

Thus for solving equations

$$71q \equiv 1\text{mod}39, \qquad 39p \equiv 1\text{mod}71.$$

we can put

$$q = 71^{24-1} = 71^{23}\text{mod}39$$

$$p = 39^{70-1} = 39^{69}\text{mod}71.$$

Then

$$71q = 71*71^{23} = 71^{24} = 1\text{mod}39$$

and similarly,

$$39p = 39*39^{69} = 39^{70} = 1\text{mod}71.$$

Let us compute $71^{23}\text{mod}39$:

$$
\begin{aligned}
q \quad &= 71^{23} \equiv (2*39-7)^{23} \equiv (-7)^{23} \equiv -7*(7^2)^{11} \equiv -7*49^{11} \\
&\equiv -7*(39+10)^{11} \equiv -7*10^{11} = -7*10*100^5 = -70*100^5 \\
&= -(2*39-8)*(3*39-17)^5 \equiv -8*17^5 = -8*17*(17^2)^2 = -136*289^2 \\
&= -(3*39+19)*(7*39+16)^2 \equiv -19*16*2 = -19*256 \\
&= -19*(39*6+22) \equiv -19*22 \\
&= -418 = -418+39*11 = 11 \mathrm{mod} 39
\end{aligned}
$$

Then

$$71q = 71*11 = 781 = 1+20*39 \equiv 1\mathrm{mod}39.$$

Hence

$$71*11-39*20 = 1.$$

Multiplying by 3 both parts of this identity we obtain:

$$213*11-117*20 = 3$$

which gives the required expression of $GCD(117,213) = 3$ as a linear combination of 117 and 213.