Sample: Abstract Algebra - Rings

Task 1. *Let R be a commutative ring with unity and let $a, b \in R$. Prove that if ab has a multiplicative inverse in R, then both a and b have multiplicative inverses.*

Proof. Let c be the multiplicative inverse of ab, so $cab = 1$. Then

$$(ca)b = 1,$$

and so

$$ca = b^{-1}$$

is the inverse of b.

Similarly, since R is commutative, $ab = ba$, and so

$$cba = cab = 1.$$

Thus

$$(cb)a = 1,$$

and therefore

$$cb = a^{-1}$$

is the inverse of a.

Task 2. *Let $R = 2\mathbb{Z}$ be the ring of even integers. Show that R contains a maximal ideal M so that R/M is not a field.*

Proof. Let $M = 4\mathbb{Z} \subset R$ be the ring of integers which are multiples of 4. We claim that R/M is not a field.

We will prove that R/M has zero divisors. Indeed, let $[2] = 2 + M$ be the class of 2 in R/M, and $[0] = M$ be the class of 0 in R/M. Then

$$[2] \cdot [2] = (2 + M)(2 + M) = 4 + M = M = [0]$$

since $4 \in M$.

Thus $[2]$ is a zero divisor in R/M, and so R/M is not a field.

Task 3. *Prove that if R is a commutative ring with unity and $f = a_n x^n + \cdots + a_0$ is a zero divisor in $R[x]$, then there exists a nonzero b in R such that*

$$ba_n = b^2 a_{n-1} = b^3 a_{n-2} = \cdots = 0.$$

Proof. The assumption that f is a zero divisor in $R[x]$ means that there exists a polynomial $g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$ such that $b_m \neq 0$ and

$$gf = 0$$

in $R[x]$.

We claim that the coefficient b_m at x_m has the required property:

$$b_m a_n = b_m^2 a_{n-1} = b_m^3 a_{n-2} = \cdots = 0.$$

Indeed, $fg = 0$ means that all the coefficients of fg are zero. Let us write exact formulas for fg:

$$gf = (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0) \cdot (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) =$$
$$(b_m a_n)\ x^{n+m} +$$
$$(b_m a_{n-1} + b_{m-1} a_n)\ x^{n+m-1} +$$

$$(b_m a_{n-2} + b_{m-1}a_{n-1} + b_{m-2}a_n)\ x^{n+m-2} + \cdots$$

Thus

$$b_m a_n = 0,$$
$$b_m a_{n-1} + b_{m-1}a_n = 0,$$
$$b_m a_{n-2} + b_{m-1}a_{n-1} + b_{m-2}a_n = 0,$$

and so on.

The first equation is what we need: $b_m a_n = 0$.

Multiplying the second equation by b_m we get:

$$0 = b_m(b_m a_{n-1} + b_{m-1}a_n) = b_m^2 a_{n-1} + b_m b_{m-1} a_n = b_m^2 a_{n-1} + b_{m-1}(b_m a_n) =$$
$$= b_m^2 a_{n-1} + b_{m-1}\cdot 0 = b_m^2 a_{n-1},$$

Thus

$$b_m^2 a_{n-1} = 0.$$

Again, multiplying the third equation by b_m^2 we obtain

$$b_m^2(b_m a_{n-2} + b_{m-1}a_{n-1} + b_{m-2}a_n) = b_m^3 a_{n-2} + b_{m-1}(b_m^2 a_{n-1}) + b_{m-2}b_m(b_m a_n) =$$
$$b_m^3 a_{n-2} + b_{m-1}\cdot 0 + b_{m-2}b_m\cdot 0 = b_m^3 a_{n-2}.$$

Thus

$$b_m^3 a_{n-2} = 0.$$

By similar arguments multiplying coefficent at x^{m+n-k} by b_m^k we will get that

$$b_m^k a_{n-k+1} = 0$$

for all k.