Sample: Math - Sentences

Question 1

Let v be a truth assignment for the sentence symbols $\{A, B, C\}$, and assume that $\bar{v} = (A \to B) \to C$. Determine which (if any) of the following must be true, and explain your reasoning.

a. $\bar{v} = A \to C$
b. $\bar{v} = \neg A \to C$

Solution.

Construct truth tables for the three expressions under consideration:

A	B	C	A->B	(A->B)->C (given as true)	A->C	˥A->C
0	0	0	1	0	1	0
0	0	1	1	1	1	1
0	1	0	1	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	0	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

It is given that the statement (A->B)->C is always true. Thus, we are to consider only the combinations of (A, B, C) in the table when this column contains "1" (highlighted by gray).
Now, consider the values in the last two columns in the lines highlighted.
(a) There is a zero in gray cells in column A->C. Thus, the statement is false.
(b) There is no one zero in gray cells in column ˥A->C. Thus, the statement is true.

Question 2
Without using a truth table, determine whether each of the following is a tautology, satisfiable but not a tautology, or unsatisfiable.

a. $\big((A \to B) \to (B \to C)\big) \to (A \to C)$
b. $(B \to (C \to A))\vee(C \to (A \to B))$
c. $(\neg A \to B)\wedge\neg(C \to A)\wedge\neg(C \to \neg B)$
d. $\neg(B \to A)\wedge(C \to A)\wedge(B \to C)$

Solution.

Use definition of implication operator: (A->B) ⇔ ˥A V B
(a) ((A->B) -> (B->C)) -> (A->C) ⇔ ((˥A V B) -> (˥B V C)) -> (˥A V B) ⇔˥ (˥(˥A V B) V (˥B V C))V (˥A V B) ⇔˥ (A^˥B V ˥B V C)V (˥A V B) ⇔˥ ((AVtrue)^˥B V C)V (˥A V B) ⇔˥ (˥B V C)V (˥A V B) ⇔ B^˥C V ˥A V B ⇔ B^(˥C Vtrue) V ˥A ⇔ B V ˥A

The expression can be true (if B = 1, for example) and false (B = 0, A = 1). So, it is satisfiable, but not a tautology.

(b) (B-> (C->A)) V (C->(A -> B)) ⇔ (˥B V (C->A)) V (˥C V (A -> B)) ⇔ (˥B V ˥C V A) V (˥C V ˥A V B) ⇔ (˥B V B) V ˥C V (A V ˥A) ⇔ true
So, it is a tautology.

(c) (┐A->B) ^ ┐(C->A) ^┐ (C->┐B) ⇔ (A V B) ^ ┐(┐C V A) ^┐ (┐ C V ┐B) ⇔ (A V B) ^ C ^ ┐A ^ C ^ B ⇔ (A V B) ^ C ^ ┐A ^ B ⇔ A ^ C ^B ^ ┐A V B^ C ^B ^ ┐A ⇔ false V C ^B ^ ┐A ⇔ C ^B ^ ┐A

The expression can be true (if B = 1 = C, A = 0) and false (B = 0). So, it is satisfiable, but not a tautology.

(d) ┐(B -> A) ^ (C->A) ^ (B-> C) ⇔ ┐(┐B V A) ^ (┐C V A) ^ (┐ B V C) ⇔ B ^ ┐A ^ (┐C V A) ^ (┐ B V C) ⇔ (┐A ^ (┐C V A)) ^ (B ^ (┐ B V C))⇔ (┐A ^ ┐C V ┐A ^ A)) ^ (B ^ ┐ B V B ^C))⇔ (┐A ^ ┐C ^ B ^C)⇔ ┐A ^ B ^ false ⇔ false
So, it is a unsatisfiable.

Question 3
Find a truth assignment satisfying the set Σ, where $\Sigma = \{(C \to B) \to (A \to \neg D),\ (B \to C) \to (D \to A), \neg(B \to \neg D)\}$, without using truth tables. Explain how you obtain the assignment.

Solution.
The last term is true, thus:
┐(B->┐D) ⇔ ┐(┐B V ┐D) ⇔ B ^D = true => B = true, D = true
The first term is true, thus:
(C -> B) -> (A -> ┐D) ⇔┐ (┐C V B) V (┐A V ┐D) ⇔ C ^ ┐B V ┐D V ┐A ⇔ C ^ false V false V ┐A ⇔ ┐A = true => A = false
The second term is true, thus:
(B -> C) -> (D -> A) ⇔┐ (┐B V C) V (┐D V A) ⇔ B ^ ┐C V ┐D V A ⇔ true ^ ┐C V false V false ⇔ ┐C = true => C = false
Thus, we get: A = false, B = true, C = false, D = true

Question 4
Prove or disprove (with a counterexample) each of the following.

a. If $(\phi \wedge \psi) | = \theta, then\ \phi | = \theta\ and\ \psi | = \theta$.
b. If $\phi | = \theta\ and\ \psi | = \theta, then\ (\phi \wedge \psi) | = \theta$.
c. If $(\phi \vee \psi) | = \theta, then\ either\ \phi | = \theta\ or\ \psi | = \theta$.
d. If either $\phi | = 0\ or\ \psi | = \theta, then\ (\phi \vee \psi) | = \theta$.
e. If $(\phi \vee \psi) | = \theta, then\ \phi | = \theta\ and\ \psi | = \theta$.

<u>Definition</u>: A set Σ of sentences is independent provided there is no $\varphi \in \Sigma$ such that $(\Sigma - \{\varphi\}) | = \varphi$.

Solution.

(a) A counterexample is: when ϕ = A, Θ = A^B, ψ = B:
(ϕ^ψ) ⇔ A^B |= Θ⇔ A^B is satisfied, but ψ|= Θ ⇔A|=A^B is not satisfied.

(b) Let ϕ |= Θ and ψ |= Θ . Then every truth assignment satisfying ϕ also satisfies Θ and every truth assignment satisfying ψ also satisfies Θ
The truth assignment satisfying (ϕ^ψ) satisfies both ϕ and ψ. Thus, it satisfies Θ. So, (ϕ^ψ) |= Θ.

(c) Let (ϕVψ) |= Θ. Then every truth assignment satisfying at least one of ϕ or ψ also satisfies Θ. Thus, for a truth assignment to satisfy Θ, it must satisfy either ϕ or ψ. So, either (ϕ |= Θ) or (ψ |= Θ) must hold.

(d) A counterexample is: when ϕ = A, Θ = B, ψ = B. In this case we get:
B|=B, thus ψ |= Θ.
(ϕVψ) = (AVB)
But it is clear that truth of the assignment AVB does not guarantee truth of the assignment B. Thus, the statement (ϕVψ) |= Θ is false.

(e) Let (ϕVψ) |= Θ. Then every truth assignment satisfying at least one of ϕ or ψ also satisfies Θ. In other words, each of the terms ϕ and ψ must guarantee the truth of Θ (because the other can be not satisfied by a statement chosen). Thus, for a truth assignment to satisfy Θ, it must satisfy both ϕ and ψ. So, (ϕ |= Θ) and (ψ |= Θ) must hold.

Question 5
Let $\Sigma = \{(A_{i+1} \rightarrow A_i)|i \geq 0\}$. Determine whether Σ is independent and prove it.

Solution.
Consider the following situations:
1 – The set contains one element only. In this case after removing the one element from the set we get an empty set left. And of course, we cannot build a non-empty set based on empty one. Thus, the set is independent.
2 – The set contains more than one term.
Assume we are to prove A(k+1) -> Ak (this is the term deleted from the set).
To do this, we need to prove some expression A(k+1) -> F from the set and then use F -> Ak to prove the expression removed (F is some statement that can include any terms A(i)).
According to the definition given the terms Ai can be found in the right part of the expressions one time only in the set. Thus, when removing A(k+1) -> Ak, there would not be any statement of the form F -> Ak, and so, we cannot prove A(k+1) -> Ak. So, the set of sentences is independent.

Question 6
(More challenging than the previous question.) Use induction to prove that every sentence in which no sentence symbol occurs more than once is satisfiable, but no such sentence is a tautology.

Solution.
For the set symbols of size one the statement is true:
Σ = A, A can be true, thus Σ is satisfiable, but A can be false, thus Σ is not a tautology
Assume that for a set of size n (n different symbols occur in Σ) the statement is true:
Σn = A1?A2?…? A(n) is satisfiable, but is not a tautology (? Means some (any) operations, the terms can be ordered randomly in the expression, parentheses are allowed between terms).
Now, consider a set of size n+1:
Σ(n+1) = A1?A2?…? A(n)?A(n+1)
(? Means some operations, the terms can be ordered randomly in the expression, parentheses are allowed between terms).
According to Boolean algebra laws, each expression has its CNF. So, the sentence obtained can be re - written as following:
Σ(n+1) = (Ai1^Ai2^…^Aik) V…V(Aj1^Aj2^…^Ajp^A(n+1))
Since each term can occur in the sentence only once, in the CNF A(n+1) will occur in one conjunct only (assume it is a last one).

According to the assumption made, any sentence with n distinct symbols is satisfiable (and it can be written in CNF too). Thus, the following sentence can be true (the same as Σ(n+1) but without A(n+1)):
Σn = (Ai1^Ai2^…^Aik) V…V(Aj1^Aj2^…^Ajp)
If there is some true conjunct except of the last one, addition of A(n+1) will not change the result and Σ(n+1) will be true as far as Σn is true.
If the last conjunct is the only one true one, we are to fix A(n+1) = true and again, we get Σ(n+1) will be true as far as Σn is true.
Thus, Σ(n+1) is satisfiable.
But when the last conjunct is the only one true one in Σn, and we fix A(n+1) = false, the last conjunct will be false and thus, Σ(n+1) will be false. So, the situation is possible for Σ(n+1) = false and thus, Σ(n+1) is not a tautology.

Question 7

(Still more challenging.) Suppose $| = (\phi \to \psi)$, where ϕ is satisfiable and where ψ is not a tautology. Show that there is a sentence σ which contains no sentence symbols other than those appearing in both ϕ and ψ such that $| = (\phi \to \sigma)$ and $| = (\sigma \to \psi)$. (Induction isn't necessarily the way to go here).

Solution.

Consider the truth table approach.
As it is given, ϕ -> ψ. Using properties of implementation, this fact means that the truth table for these expressions can contain only lines of the following type:

Parameters set	ϕ	ψ
….	0	0
	0	1
	1	1

Thus, the lines in the truth table can be re-ordered and represented as following:

{… combination of symbols used in ϕ and ψ …}	ϕ	ψ
C1	1	1
…	…	…
Cn	1	1
C(n+1)	0	1
…	…	…
Cm	0	1
C(m+1)	0	0
…	…	…

Where m and n are some positive integers or zero m >= n.
Now, we can add one column to this table, which will have some number of "1" between m and n:

{… combination of symbols used in ϕ and ψ …}	ϕ	ψ	σ
C1	1	1	1
…	…	…	…
Cn	1	1	1
C(n+1)	0	1	1
…		…	…
Ct		1	1
C(t+1)		1	0
…	…	…	…
Cm	0	1	0

C(m+1)	0	0	0
…	…	…	…

Where t is some positive integer or zero, m >= t >= n. It is clear, the table and value of t above will exist for each m >= n.

Also, the table indicates that σ -> ψ and ϕ -> σ (according to definition of implication).

Now we need to build a formula for σ as a function of {C1, C2, …} using the truth table (we can use Karnaugh maps to do it). The formula will contain only the symbols presented in ψ and ϕ and will satisfy both σ -> ψ and ϕ -> σ.